family. The dual linear program of (1) is

$$
\begin{aligned}
\text{min.} \quad & \sum y_m \\
\text{s.t.} \quad & \sum_{m \ni e} y_m \geq 1 \quad \text{for each hyperedge } e \\
& y_m \geq 0 \qquad \text{for all matching } m
\end{aligned}
\tag{2}
$$

A solution to this linear program, when $n = k \cdot r$, is

$$
y_m = ((k-1)!)^{-1} \prod_{i=1}^{k-1} \binom{n - ir}{r} = \frac{((k-1) \cdot r)!}{(r!)^{k-1} \cdot (k-1)!}
$$

for every m. This the desired upper bound of $\binom{n-1}{r-1}$ to (1).

3 Proving Turán's Theorem

Tighten up the LP's by adding constraints obtained from smaller instances.

We first write the following "naïve" LP relaxation, denoted by P_{r+1}, considering Turán's theorem as a maximization problem on complete graphs.

$$
\begin{aligned}
\text{max.} \quad & \sum x_e \\
\text{s.t.} \quad & \sum_{e \in t} x_e \leq \binom{r+1}{2} - 1 \quad \text{for each complete subgraph } K_{r+1}\ t \\
& x_e \geq 0 \qquad \text{for each edge } e
\end{aligned}
\tag{3}
$$

The optimal solution of (3) can be shown via duality to be assigning $x_e = 1 - 1/\binom{r+1}{2}$ for each edge, which gives an objective of $\left(1 - 1/\binom{r+1}{2}\right)\binom{n}{2}$.

Recall that Turán's theorem asserts an upper bound of $(1-1/r) \cdot n^2/2$, which is lower than our above result.

We are going to approach this bound by enhancing the linear program (3) iteratively.

Consider instantiating (3) with $n = r+2$, the optimal, as we have discussed, would be

$$
s_{r+1} = \left(1 - 1/\binom{r+1}{2}\right)\binom{r+2}{2} \qquad = \frac{(r+2)(r-1)}{(r+1)r} \cdot \frac{(r+2)(r+1)}{2}
$$

This bound s_{r+1} exactly matches Turán's assertion

$$
s_{r+1} = (1 - 1/r) \cdot (r+2)^2/2 \tag{4}
$$

Now we add a new constraint to (3):

$$
\sum_{e \in t} x_e \leq \lfloor s_{r+1} \rfloor \qquad \text{for each complete subgraph } K_{r+2} \tag{5}
$$

Denote the linear program after addition of (5) by P_{r+2}

Intuitively, $x_e = \lfloor s_{r+1} \rfloor / \binom{r+2}{2}$ for all edges e would be an optimal solution to P_{r+2}. Indeed, this can be routinely proved by duality.

Denote the optimal objective value of P_{r+2} with $n = r+3$ by s_{r+2}, and recursively define the linear program after addition of

$$\sum_{e \in t} x_e \leq \lfloor s_i \rfloor \qquad \text{for each complete subgraph } K_{i+1} \tag{6}$$

to LP P_i by P_{i+1}, and similarly denote the optimal value of P_i by s_i.

Claim 3.1. *For $n = i+1$ the bound we get from P_i is as good as Turán's, or*

$$\lfloor s_i \rfloor \leq \frac{r-1}{r}\frac{i^2}{2}$$

Proof. We prove by induction on i.

1. The base case when $i = r+1$ has already been established.
2. The inductive step is equivalent to, assuming

$$\lfloor s_i \rfloor \leq \frac{r-1}{r}\frac{i^2}{2}$$

prove

$$\left\lfloor \lfloor s_i \rfloor / \binom{i+1}{2} * \binom{i+2}{2} \right\rfloor \leq \frac{r-1}{r}\frac{(i+2)^2}{2}$$

We prove by two cases:

(a) i is not a multiple of r, let $i = q \cdot r + k$ where $0 < k < r$ is the remainder of i divided by r.

$$\begin{aligned} & \lfloor s_i \rfloor / \binom{i+1}{2} * \binom{i+2}{2} \\ \leq & \left\lfloor \frac{r-1}{r} \cdot \frac{i^2}{2} \right\rfloor \cdot \frac{i+2}{i} \\ = & \left\lfloor (r-1)\frac{(rq+k)^2}{2} \right\rfloor \\ = & \left\lfloor (r-1)qk + \frac{(r-1)rq^2}{2} + k^2 \cdot \frac{r-1}{2r} \right\rfloor \end{aligned}$$

(b) i is a multiple of r.

□

4 weighted triangle-free subgraph

The problem of maximum triangle-free subgraph looks for a maximum edge subset of input graph G without a triangle. The problem is proven NP-Hard [papadimitriou 198x] and APX-Hard [Kortsarz 2008].

References

[1] Noga Alon and Joel H. Spencer. *The Probabilistic Method, with an appendix on open problems by Paul Erdős.* John Wiley & Sons, 1992.