When is an Operator Diagonalizable?

Ex Recall example when $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to$ cannot be diagonalized

$P_A(\lambda) = \lambda^2 = 0 \Rightarrow m_a(0) = 2$

$\xi_1 = (1, 0)^T$ is the only eigenvector $\Rightarrow m_g(0) = 1$

Problem We cannot construct an eigenvector matrix P s.t. $A = P\Lambda P^{-1}$ since aren't enough eigenvectors

$$\boxed{\text{Root of the evil: There exist eigenvalue } \lambda \text{ s.t. } m_a(\lambda) \neq m_g(\lambda)}$$

Th 4.6. Let $\{\xi_1 \dots \xi_m\}$ be eigenvectors of L corresponding to <u>unequal</u> eigenvalues $\{\lambda_1 \dots \lambda_m\}$. Then $\{\xi_1 \dots \xi_m\}$ are linearly independent.

▼ Suppose that ξ_i's were linearly dependent.

And r is the smallest integer s.t. $\{\xi_1 \dots \xi_r\}$ are lin. dep.

That is $a_1\xi_1 + \dots + a_r\xi_r = 0$ where $\{a_i\}$ are not all zero.

but $b_1\xi_1 + \dots + b_{r-1}\xi_{r-1} \neq 0$ (so if $\sum_{i=1}^{r-1} b_i\xi_i = 0$ iff $b_i = 0$ for all $i = \overline{1, r-1}$)

Apply operator $(L - \lambda_r I_r)$: $\sum_{i=1}^{r} a_i\xi_i = 0 \Rightarrow$

$$0 = (L - \lambda_r I)\sum_{i=1}^{r} a_i\xi_i = \sum_{i=1}^{r} a_i(L - \lambda_r I)\xi_i = \sum_{i=1}^{r} a_i(L\xi_i - \lambda_r I\xi_i) =$$

$$= \sum_{i=1}^{r} a_i(\lambda_i\xi_i - \lambda_r\xi_i) = \sum_{i=1}^{r} a_i(\lambda_i - \lambda_r)\xi_i = \sum_{i=1}^{r-1} a_i(\lambda_i - \lambda_r)\xi_i$$

because $(\lambda_r - \lambda_r) = 0$ and can be deleted from the sum.

Sin $\{\lambda_1 \dots \lambda_m\}$ are all different $\Rightarrow a_i(\lambda_i - \lambda_r) \neq 0$ for all $i = \overline{1, r-1}$ and at least one $a_i \neq 0 \Rightarrow \{\xi_i\}_{i=1}^{r-1}$ lin. dep. Contradiction! ▲

Corollary 4.7 If the characteristic polynomial $p_L(\lambda)$ has n distinct roots, where $L: V \to V$, and $n = \dim(V)$, then L is diagonalizable.

▼ if $p_L(\lambda)$ has n distinct roots $\Rightarrow \exists \ \{\xi_i\}_{i=1}^{n}$ eigenvectors for each λ_i $i=\overline{1,n}$, $L\xi_i = \lambda_i \xi_i$, and by previous theorem $\{\xi_i\}_{i=1}^{n}$ linear independent set then transition matrix exist

$$P = (\xi_1 \xi_2 \dots \xi_n) \Rightarrow P^{-1} \exists \quad (\det P \neq 0)$$

$$(L)_\xi = P \Lambda P^{-1}$$

▲

Th 4.8 If λ_0 is an eigenvalue of L then

$$1 \leq m_g(\lambda_0) \leq m_a(\lambda_0).$$

▼ Since λ_0 is an eigenvalue $\Rightarrow$ it has an eigenvector and $m_g(\lambda_0)$ at least equal 1.

① Need a fact about matrices (p 79 Jadun)

If a matrix A takes a block diagonal form then

$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where A, C are square matrices and B, 0 are possibly rectangular matrices.

$$\Rightarrow \boxed{P_M(\lambda) = P_A(\lambda)\, P_C(\lambda)}$$

that is characteristic polynomial does not depend on B at all!

- Let λ_0 be an eigenvalue with eigenvectors $\xi_1 \dots \xi_M$ where $M = m_g(\lambda_0)$
  create eigenspace of λ_0: $E_{\lambda_0} = \text{span}\{\xi_1 \dots \xi_M\}$
  $\dim(E_{\lambda_0}) = M = m_g(\lambda_0)$
- ∴ add vectors $\{\eta_1 \dots \eta_{n-m}\}$ to form a basis in V

$\mathcal{D} = \{\xi_1 \dots \xi_m, \eta_1 \dots \eta_{n-m}\}$ basis in V

∴ $$\begin{cases} L\xi_1 = \lambda_0 \xi_1 \\ \vdots \\ L\xi_m = \lambda_0 \xi_m \\ L\eta_1 = \sum_{i=1}^{m} \alpha_i^{(1)} \xi_i + \sum_{j=1}^{n-m} \beta_j^{(1)} \eta_j \\ \vdots \\ L\eta_{n-m} = \sum_{i=1}^{m} \alpha_i^{(n-m)} \xi_i + \sum_{j=1}^{n-m} \beta_j^{(n-m)} \eta_j \end{cases}$$

$$[L]_{\mathcal{D}} = \begin{pmatrix} | & & | & | & \\ L\xi_1 & \dots & L\xi_m & L\eta_1 & \dots \\ | & & | & | & \end{pmatrix}$$

$$(L)_{\mathcal{D}} = \begin{pmatrix} \lambda_0 & 0 & 0 & \alpha_1^{1} & \dots & \alpha_1^{n-m} \\ 0 & \lambda_0 & \vdots & & & \\ 0 & 0 & \ddots & & & \\ & & \lambda_0 & \alpha_m^{(1)} & \dots & \alpha_m^{n-m} \\ \vdots & \vdots & \vdots & \beta_1^{1} & \dots & \beta_1^{n-m} \\ & & & \vdots & & \vdots \\ 0 & 0 & 0 & \beta_{n-m}^{(1)} & \dots & \beta_{n-m}^{(n-m)} \end{pmatrix} = \begin{pmatrix} \lambda_0 I_m & A \\ \mathbb{O} & B \end{pmatrix}$$

$\lambda_0 I_m$: $m_g(\lambda_0) \times m_g(\lambda_0)$
A : $m_g(\lambda_0) \times (n - m_g(\lambda_0))$
B : $(n - m_g(\lambda_0)) \times (n - m_g(\lambda_0))$

$$P_L(\lambda) = P_{[L]_{\mathcal{D}}}(\lambda) = (\lambda - \lambda_0)^{m_g(\lambda)} P_B(\lambda)$$

Since $P_L(\lambda)$ is divisible by $(\lambda-\lambda_0)^{m_g(\lambda_0)}$ it follows that the algebraic multiplicity of λ_0 is at least as large as geometric multiplicity.

$$\boxed{1 \le m_g(\lambda_0) \le m_a(\lambda_0)}$$

▲

Example

$L: V \to V$, $\dim(V) = 4$

λ_0 is eigenvalue L with $m_g(\lambda_0) = 2$

$L d_1 = \lambda_0 d_1$

$L d_2 = \lambda_0 d_2$

Define a new basis $\{d_1\, d_2\, d_3\, d_4\}$ where d_1, d_2 are eigenvectors d_3 & d_4 any other pair of vectors s.t. $\{d_i\}_{i=1}^{4}$ are lin indep. set of vectors.

$$\Rightarrow \quad (L)_D = \begin{pmatrix} \lambda_0 & 0 & a_{11} & a_{12} \\ 0 & \lambda_0 & a_{21} & a_{22} \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{12} & b_{22} \end{pmatrix} \equiv \begin{pmatrix} \lambda_0 I_2 & A \\ 0 & B \end{pmatrix}$$

v vector in V $\quad v = \underbrace{\alpha_1 d_1 + \alpha_2 d_2}_{\text{in } E_{\lambda_0}} + \underbrace{\beta_1 d_3 + \beta_2 d_4}_{\text{possibly projects into } E_{\lambda_0}} \Rightarrow (V)_D = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix} \equiv ($

Block multiplication:

$$(L)_D (v)_D = \begin{pmatrix} \lambda_0 I_2 & A \\ 0 & B \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \lambda_0 I_2 \alpha + A\beta \\ \emptyset \alpha + B\beta \end{pmatrix} = \begin{pmatrix} \lambda_0 \alpha_1 + a_{11}\beta_1 + a_{12} \\ \lambda_0 \alpha_2 + a_{21}\beta_1 + a_{22} \\ b_{11}\beta_1 + b_{12}\beta \\ b_{21}\beta_1 + b_{22}\beta \end{pmatrix}$$

(1) $(L)_D$ only "stretches/rotates" by λ_0 along d_1 & d_2

(2) $(L)_D$ does not mix components of d_1 & d_2 into $d_3\, d_4$ directions

(3) reflects possible non-zero projections of $d_3\, d_4$ onto E_{λ_0} subspace

Th 4.9 An operator L on a complex vector space is diagonalizable iff all of its eigenvalues have $\boxed{m_a = m_g.}$

(1) If $m_a = m_g \Rightarrow L$ is diagonalizible

- let $\xi_1 \dots \xi_{m_g(\lambda_1)}$ be a basis for E_{λ_1} -eigenspace corresponding to λ_1

def If λ is an eigenvalue, the set of ξ's such that $L(\xi) = \lambda\xi$ is called the eigenspace corresponding to λ, E_λ

$\xi_{m_g(\lambda_1)+1} \dots \xi_{m_g(\lambda_1)+m_g(\lambda_2)}$ - basis for E_{λ_2}

and so on $\sum_{i=1}^{K} m_g(\lambda_i) = \sum_{i=1}^{K} m_a(\lambda_i) = n$

if set $\{\xi\}_{i=1}^{n}$ are lin. Ind. $\Rightarrow$ its a basis in V and L is diagonalizible (P^{-1} exist)

To see linear independence assume that it's not true

$$\Rightarrow \quad 0 = a_1\xi_1 + \dots + a_n\xi_n = \underbrace{a_1\xi_1 + \dots + a_{m_g(\lambda_1)}\xi_{m_g(\lambda_1)}}_{d_1} + \underbrace{\dots}_{d_2} =$$

The sum of first $m_g(\lambda_1)$ terms $\equiv d_1$

The sum of second $m_g(\lambda_2)$ $\equiv d_2$ and so on

$\Rightarrow \quad 0 = \sum_{i=1}^{K} d_i \qquad d_i$ is $\in E_{\lambda_i} \Rightarrow d_i$ is eigenvector corresponding λ_i

λ_i are all different, hence by th 4.6 d_i are linearly independent $\Rightarrow d_i = 0$ for $i = \overline{1K}$

However $d_i = \sum a_i\xi_i$ where ξ_i basis of $E_{\lambda_i} \Rightarrow d_i \neq 0$ only if $a_i = 0$ for all $i \Rightarrow a_i = 0$ and set $\{\xi\}_{i=1}^{n}$ is lin. Ind.

② If L is diagonalizable $\Rightarrow m_a(\lambda_i) = m_g(\lambda_i)$

$L = P \Lambda P^{-1}$ Λ - diagonal $P = \begin{pmatrix} | & & | \\ \xi_1 & \dots & \xi_n \\ | & & | \end{pmatrix}$, ξ_i - e.vec

$\Rightarrow \det P \neq 0$ and $\Rightarrow \{\xi_1 \dots \xi_n\}$ are lin indep.

for eigenvalue i we have $m_g(\lambda_i)$ eigenvectors ξ

hence $\sum_{i=1}^{K} m_g(\lambda_i) = n \quad / - \sum_{i=1}^{K} m_a(\lambda_i)$

$$\sum_{i=1}^{K} m_g(\lambda_i) - m_a(\lambda_i) = \underbrace{n - \sum_{i=1}^{K} m_a(\lambda_i)}_{} = 0$$

by the ~~th~~ characteristic polynomial has n-roots.

Hence $m_g(\lambda_i) = m_a(\lambda_i)$ ▲

Example L is an operator in 8 dim space.

$\lambda_1 = 5 \leftrightarrow m_a(\lambda_1) = m_g(\lambda_1) = 3$

$\lambda_2 = 6 \leftrightarrow m_a(\lambda_2) = m_g(\lambda_2) = 2$

$\lambda_3 = -2 \leftrightarrow m_a(\lambda_3) = m_g(\lambda_3) = 3$

$E_1 = \text{span}\{\xi_1, \xi_2, \xi_3\}$

$E_2 = \text{span}\{\xi_4, \xi_5\}$ $\rightarrow P = \begin{pmatrix} | & | & & | \\ \xi_1 & \xi_2 & \dots & \xi_8 \\ | & | & & | \end{pmatrix}$

$E_3 = \text{span}\{\xi_6, \xi_7, \xi_8\}$

$(L)_B = P \Lambda P^{-1}$ where $\Lambda = \begin{pmatrix} 5I_3 & & 0 \\ & 6I_2 & \\ 0 & & -2I_3 \end{pmatrix}$

Simultaneous Diagonalization of Two Operators

?: Given 2 operators, is there a single eigenvector basis that diagonalizes them both?

A: Yes, under special circumstances (commuting operators)

Def Two operators **commute** if $AB = BA$

Notation The commutator bracket $[A, B] \equiv AB - BA$

(A and B commute if $[A, B] = 0$)

Th 4.10 Two diagonalizable operators on a finite dimensional vector space are simultaneously diagonalizable iff they commute.

▼ Assume A & B are diagonilizable operators in n-dim Vector space V AND A & B simultaneously diagonalizable ⇒ let's show that they commute that is $AB = BA$

i) if A & B simultaneously diagonalizable it means that $\exists$ basis $\mathcal{B} = \{b_i\}_{i=1}^{n}$ s.t. $Ab_i = \lambda_i b_i$ & $Bb_i = \mu_i b_i$

$\Rightarrow ABb_i = A(Bb_i) = A\mu_i b_i = \mu_i (Ab_i) = \mu_i \lambda_i b_i =$

$= \lambda_i (\mu_i b_i) = \lambda_i B b_i = B \lambda_i b_i = BAb_i$

Since AB acting on any basis vector is the same as BA acting on that basis, AB acting on any linear combination of basis vectors, i.e. on $\forall v \in V$, is the same as BA. So $\boxed{AB = BA}$.

So if A, B are simulteneously diagonazable they must commute! $\boxed{AB = BA}$

Lemma 4.11 If $[A, B] = 0$ then B maps each eigenspace of A into itself.

▼ Suppose $|v\rangle$ is an eigenvector of A with eigenvalue λ, t.is $A|v\rangle = \lambda|v\rangle$

We will show that $B|v\rangle$ is the same eigenspace $E_\lambda(A)$

$$A(B|v\rangle) = (AB)|v\rangle = (BA)|v\rangle = B\lambda|v\rangle = \lambda(B|v\rangle)$$

$$\therefore \quad \boxed{B|v\rangle \in E_\lambda(A)}$$

So $B|v\rangle$ is also in the eigenspace of A.

the rest of the proof that is if $AB = BA$ and A & B diagonalizable $\Rightarrow$ A & B simult. diag. read p 83 in Sadun.

? Simultaneous diagonalization of A & B: Method

(i) Find eigenvectors/eigenvalues of A $\rightarrow$ call this eigenvector basis $\{d_1 \ldots d_n\} = \mathcal{D}$

(ii) This eigenvectors of A may not be automatically eigenvectors of B, we make use of the fact that $B: E_\lambda(A) \rightarrow E_\lambda(A)$ and express $(B)_\mathcal{D}$, the matrix B in the $\mathcal{D}$ basis

$$(B)_{\mathcal{D}} \equiv (\tilde{\Lambda}_B)_{\mathcal{D}} = P_{\mathcal{D}E}(B)_E P_{E\mathcal{D}} = P_{E\mathcal{D}}^{-1}(B)_E P_{E\mathcal{D}}$$
$$(A)_{\mathcal{D}} \equiv \Lambda_A = P_{\mathcal{D}E}(A)_E P_{E\mathcal{D}} = P_{E\mathcal{D}}^{-1}(A)_E P_{E\mathcal{D}}$$

$$\therefore \boxed{\begin{aligned}(B)_E &= P_{E\mathcal{D}}(B)_{\mathcal{D}} P_{E\mathcal{D}}^{-1}\\ (A)_E &= P_{E\mathcal{D}}(\Lambda_A) P_{E\mathcal{D}}^{-1}\end{aligned}} \qquad \boxed{\begin{aligned}(B)_{\mathcal{D}} &= P_{E\mathcal{D}}^{-1}(B)_E P_{E\mathcal{D}}\\ (\Lambda_A) &= P_{E\mathcal{D}}^{-1}(A)_E P_{E\mathcal{D}}\end{aligned}}$$

$$P_{E\mathcal{D}} = \begin{pmatrix} | & & | \\ d_1 & \cdots & d_n \\ | & & | \end{pmatrix}$$ - eigenvectors of A in basis E

$$\Lambda_A = \begin{pmatrix} \lambda_1 I_{m_g(\lambda_1)} & & & 0 \\ & \lambda_2 I_{m_g(\lambda_2)} & & \\ & & \ddots & \\ 0 & & & \lambda_n I_{m_g(\lambda_n)} \end{pmatrix}$$

diagonal matrix consisting of eigenvalues of A.

(iii) Because B maps eigenspace of A into themselves it must be that

$$(B)_{\mathcal{D}} = \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_m \end{pmatrix}$$

assuming m distinct eigenvalues of (A amongst n)
each B_j ($j=\overline{1,m}$) is a $m_g(\lambda_j)\times m_g(\lambda_j)$ block.

This form of $(B)_{\mathcal{D}}$ express the fact that
$B: E_{\lambda_j}(A) \to E_{\lambda_j}(A)$ for every $j=\overline{1,m}$

(iv) We can diagonalize $(B)_{\mathcal{D}}$ by calcul. eigenpairs of $B_1 \ldots B_m$ separately. $\Rightarrow$

$\Rightarrow$ if $b_1 \in E_{\lambda_1}(B_1) \Rightarrow \begin{pmatrix} b_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in E_{\lambda_1}((B)_D)$

$b_2 \in E_{\lambda_2}(B_2) \Rightarrow \begin{pmatrix} 0 \\ b_2 \\ \vdots \\ 0 \end{pmatrix} \in E_{\lambda_2}((B)_D)$

- Diagonalizing $(B)_D$ means finding n linearly independent eigenvectors that diagonalize $(B)_D$ (and thus B)

- Since these $(B)_D$ eigenvectors live within corresponding eigenspaces of A, the eigenvectors of $(B)_D$ diagonalize both A & B

i.e. the eigenvectors of $(B)_D$ must also be eigenvectors of the original matrix $(A)_E$

Example

$$A = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = BA = \begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$p_A(\lambda) = \det \begin{pmatrix} -\lambda & 4 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 4 \\ 0 & 0 & 1 & -\lambda \end{pmatrix} = (-\lambda)^2(\lambda^2-4) - 4(\lambda^2-4) =$$

$$= (\lambda^2-4)(\lambda^2-4) = (\lambda+2)^2(\lambda-2)^2 \qquad \lambda_{1,2} = 2, \quad \lambda_{3,4} = -2$$

eigenvectors:

$$\lambda_{1,2} = 2 \qquad \begin{pmatrix} -2 & 4 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} x_1 - 2x_2 = 0 \\ x_3 - 2x_4 = 0 \end{matrix}$$

$\lambda_{1,2} = 2 \quad d_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad d_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$

$\lambda_{3,4} = -2 \quad \begin{pmatrix} 2 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad d_3 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad d_4 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}$

Let's write $(B)_D$: $\quad P_{ED} = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & -1 \end{pmatrix}$

$P_{ED}^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix}$

$\therefore (B)_D = \frac{1}{4}\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 2 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & -1 \end{pmatrix}$

$\Rightarrow (B)_D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Now we need to find eigenvectors of $(B)_D$.

$P_B(\lambda) = \det\begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{pmatrix} = (\lambda^2(\lambda^2-1)) - 1(\lambda^2-1) =$
$= (\lambda^2-1)(\lambda^2-1) = (\lambda-1)^2(\lambda+1)^2$

E. vectors $\lambda = 1$: $\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow b_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad b_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = -1$: $b_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad b_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$

New basis $\mathcal{B} = \{b_1 \dots b_4\}$ is expressed w/r to the $\mathcal{D} = \{d_1 \dots d_4\}$ basis of e.vect. of A

$$\left.\begin{aligned} b_1 &= d_1 + d_2 \\ b_2 &= d_3 + d_4 \\ b_3 &= d_1 - d_2 \\ b_4 &= d_3 - d_4 \end{aligned}\right\} \begin{aligned} Bb_1 &= 1 b_1 , & Ab_1 &= A(d_1+d_2) = 2d_1 + 2d_2 = 2 b_1 \\ Bb_2 &= 1 b_2 , & Ab_2 &= A(d_3+d_4) = -2d_3 - 2d_4 = -2 b_2 \\ Bb_3 &= -1 b_3 , & Ab_3 &= A(d_1-d_2) = 2d_1 - 2d_2 = 2 b_3 \\ Bb_4 &= -1 b_4 , & Ab_4 &= A(d_3-d_4) = -2d_3 + 2d_4 = -2 b_4 \end{aligned}$$

$$\therefore \quad \begin{aligned} Bb_1 &= b_1 & Ab_1 &= 2b_1 \\ Bb_2 &= b_2 & Ab_2 &= -2b_2 \\ Bb_3 &= -b_1 & Ab_3 &= 2b_3 \\ Bb_4 &= -b_4 & Ab_4 &= -2b_4 \end{aligned}$$

$\mathcal{B} = \{b_1 \dots b_4\}$ simultaneously diagonalises A & B

$$\begin{aligned} b_1 &= d_1 + d_2 = (2\ 1\ 2\ 1)^T \\ b_2 &= d_3 + d_4 = (2\ {-1}\ 2\ {-1})^T \\ b_3 &= d_1 - d_4 = (2\ 1\ {-2}\ {-1})^T \\ b_4 &= d_3 - d_4 = (2\ {-1}\ {-2}\ 1)^T \end{aligned}$$

In the new basis $(B)_{\mathcal{B}} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ $(A)_{\mathcal{B}} = \begin{pmatrix} 2 & & & 0 \\ & -2 & & \\ & & 2 & \\ 0 & & & -2 \end{pmatrix}$

Simultaneous diagonalisation of functions of A & B

If we have found a basis $\{d_1 \ldots d_n\} = \mathcal{D}$ that simultaneously diagonalizes A & B

$$A d_i = \lambda_i d_i \quad \& \quad B d_i = \mu_i d_i \qquad i = \overline{1,n}$$

$\Rightarrow$ $\mathcal{D} = \{d_1 \ldots d_n\}$ simultaneously diagonalizes "reasonable functions" of A, B as well!

Examples (i) $C = A + B \rightarrow (A+B) d_i = (\lambda_i + \mu_i) d_i$

$$C d_i = (\lambda_i + \mu_i) d_i$$

$\therefore$ C is also diagonalised by $\mathcal{D}$ with e.values $(\lambda_i + \mu_i)$

i.e.
$$(C)_{\mathcal{D}} = \begin{pmatrix} (\lambda_1 + \mu_1) I_{m_g}(\lambda_1) & & \bigcirc \\ & \ddots & \\ \bigcirc & & (\lambda_m + \mu_m) I_{m_g}(\lambda_m) \end{pmatrix}$$

(ii) $C = AB \rightarrow (AB) d_i = A \mu_i d_i = \mu_i A d_i = \mu_i \lambda_i d_i$

$\Rightarrow$ Eigenvalues of (AB) $\lambda_i \mu_i$

Eigenvectors $\mathcal{D} = \{d_1 \ldots d_m\}$

(iii) $C = (A+B)^n \rightarrow (A+B)^n d_i = (A+B)^{n-1}(A+B) d_i =$

$$= (\lambda_i + \mu_i)(A+B)^{n-1} d_i = (\mu_i + \lambda_i)^n d_i$$

$$\Rightarrow (A+B)^n d_i = (\lambda_i + \mu_i)^n d_i$$