Appendix I. Bob Smith's Phase Plug

In his 1953 paper [1] Smith describes the behaviour of the compression cavity of a compression driver as a boundary value problem [2]. This allows him to derive phase-plug channel arrangements required in order to avoid excitation of radial modes in the compression cavity. In this appendix the channel geometry conditions suggested by Smith are re-derived.

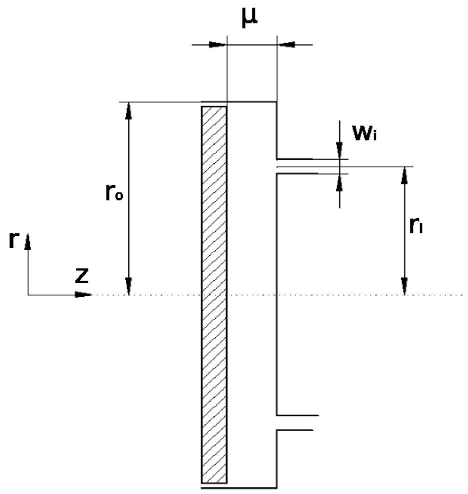

Figure 1: Cylindrical representation of compression driver

Smith suggests that the compression cavity, being only slightly curved, may be analysed, with only small error, in a cylindrical co-ordinate system. Figure 1 shows the approximated cavity in cylindrical coordinates with a single exit channel, i.

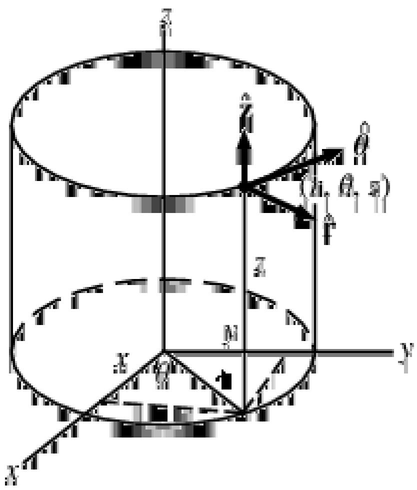

Figure 2: Cylindrical coordinate system notation.

The cylindrical coordinate system in this work is given in the notation of Beyer [3] as depicted in Figure 2. We start from the homogeneous Helmholtz equation [4 p.18] in equation 1 and first consider the acoustical behaviour of the undriven rigid-walled compression cavity.

$$\nabla^2 p + \frac{\omega^2}{c_o^2} p = 0 \qquad (1)$$

In cylindrical coordinates the Laplacian can be written as shown below in equation 2 [5].

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \qquad (2)$$

Solutions can be found using the method of separation of variables [4 p.116][6].

$$p(r,\theta,z,t) = R(r)\Theta(\theta)Z(z)T(t) \qquad (3)$$

$$T(t) = T_1 e^{-j\omega t} + T_2 e^{j\omega t} \qquad (4)$$

$$Z(z) = Z_1 e^{-jk_z z} + Z_2 e^{jk_z z} \qquad (5)$$

$$\Theta(\theta) = \Theta_1 e^{-jm\theta} + \Theta_2 e^{jm\theta} \qquad (6)$$

$$R(r) = R_1 J_m(k_r r) + R_2 Y_m(k_r r) \qquad (7)$$

With T_1, T_2, Z_1, Z_2, Θ_1, Θ_2, R_1 & R_2 as arbitrary constants.

In order to effectively apply this general solution of the wave equation to the situation at hand we first make some simplifications.

Selecting a convention for time dependence, we set $T_2=0$.

The compression cavity is totally axisymmetric in both geometry and excitation. We are not interested in pressure variations in θ. In the driven case, the circumferential function $\Theta(\theta)$ has only trivial solutions in our frequency band of interest and so we set $m=0$, $\Theta_1+\Theta_2=1$.

The compression cavity is small in z and in the frequency band of interest only trivial behaviour is experienced. We set $k_z=0$, $Z_1+Z_2=1$.

The Bessel function of the second kind $Y_m(k_r r)$, appearing in the radial function, is singular at $k_r r=0$. Restraining our solutions to be finite in the cavity we set $R_2=0$.

Our simplified solution to the wave equation in the rigid-walled compression cavity is.

$$p(r,t) = A_1 J_0(k_r r) e^{-j\omega t} \qquad (8)$$

To this solution we prescribe the conditions for the rigid boundary at the outside diameter of the cavity, r_0 .

$$\frac{dp}{dr} = 0 \Big|_{r=r_0} \qquad (9)$$

To resolve this condition we require that

$$\frac{d\,J_0\left(k_r r\right)}{dr}=0\Big|_{r=r_0} \tag{10}$$

Values of k_r that satisfy equation 10 are found from the roots of the Bessel function of the first kind J_1 [7, equation 62].

$$J_1(j_n)=0 \quad , \quad k_n=\frac{j_n}{r_0} \tag{11}$$

We find that the homogeneous wave equation in the rigid walled compression cavity has solutions only at district values of k_n. These values of k_n give the eigenvalues of the system.

$$\omega_n=k_n c_0 \tag{12}$$

The eigenfunctions of the cavity are given by inserting the values of k_n into the expression for the spatial pressure variation in the cavity from equation 8.

$$\Psi(r)=A_1 J_0\left(k_n r\right) \tag{13}$$

The normalisation term A_1 in equation 13 is chosen to satisfy equation 14.

$$\int_V \Psi(r)^2 dV=V \tag{14}$$

I.i. Driven Behaviour Of The Cavity

The pressure in a undamped, driven acoustical cavity excited by motion of it's walls can be described in terms of the rigid walled eigenfunctions and eigenfrequencies.

$$p(\boldsymbol{x})=\sum_{n=0}^{\infty}\frac{j\omega\rho_o\Psi_n(\boldsymbol{x})}{V\left[k_n^2-k^2\right]}\int_s \Psi_n(\boldsymbol{y})u(\boldsymbol{y}).\boldsymbol{n}\,dS \tag{15}$$

Derivation of this expression is clearly described in [8, pp. 314-315] and [4, pp. 277-279]. The choice of normalisation, equation 14, is important for the use of this expression [8, p.313].

I.i.i. Behaviour Without Exit Channels

We first will look at the case were the cavity is driven only by the membrane on the entrance side of the compression cavity with no channels on the exit side of the cavity. The surface velocity is zero at all locations on the enclosure walls except for the surface occupied by the membrane. We need only perform the integral over this part of the surface.

$$\int_S \Psi_n(\boldsymbol{y})u(\boldsymbol{y}).\boldsymbol{n}\,dS=\int_{r=\cdot}^{r_o}\Psi_n(r)u_o\,dS \tag{16}$$

This reveals an important detail. The eigenfunctions form an orthonormal set obeying the orthogonality relation shown below.

$$\int_V \Psi_n(\boldsymbol{y})\Psi_m(\boldsymbol{y})\,dV = V\,\delta_{nm} \qquad (17)$$

For this particular case, thin cylindrical cavity and eigenfunctions without variation across the thickness of the cavity (z direction), this expression can be re-written.

$$\int_V \Psi_n(\boldsymbol{y})\Psi_m(\boldsymbol{y})\,dV = \int_{r=0}^{r_o}\int_{z=0}^{\mu} \Psi_n(\boldsymbol{y})\Psi_m(\boldsymbol{y})\,dS\,dz$$
$$\rightarrow \int_{r=0}^{r_o} \Psi_n(\boldsymbol{y})\Psi_m(\boldsymbol{y})\,dS = \frac{V\,\delta_{nm}}{\mu} = S\,\delta_{nm} \qquad (18)$$

The normal velocity of the radiating membrane is constant over the integral in equation 16 and so it is the same distribution as the zeroth eigenfunction. Using 18 we are able to evaluate the integral in equation 16 giving.

$$\int_{r=0}^{r_o} \Psi_n(r)\,u_o\,dS = u_o S\,\delta_n \qquad (19)$$

This indicates that only the zeroth mode contributes to the pressure variation in the compression cavity in this case. Equation 15 becomes simply.

$$p = \frac{\rho_o c_o^2}{V}\frac{S u_o}{j\omega} \qquad (20)$$

This expression is familiar, the pressure variation in the cavity is that of a piston driving a theoretical ideal acoustical compliance.

This result has an interesting implication, using this representation of a compression driver, all excitation of the acoustical modes in the compression cavity occurs due to the presence of the exit channels and specifically the motion of air at their entrances.

I.i.ii. Behaviour With Exit Channels

The arrangement of a compression driver is to have not only a radiating membrane on one face of the cylindrical compression cavity but also a number of exit channels on the opposite face, which lead through the phase plug and to the horn, through which sound is radiated.

The exit side surface normal acoustic velocity is represented by the function $u_e(r)$ defined as shown below in equation 21.

$$u_e(r) = w_1 u_1\,\delta(r - r_1) + w_2 u_2\,\delta(r - r_2) + \ldots + w_N u_N\,\delta(r - r_N) \qquad (21)$$

Here we make use of the Dirac delta function [9] to define that the normal velocity on the exit side of the cavity is only non-zero at discrete values of r where the annular phase plug channels meet the cavity. The channels are assumed to be narrow enough so that they can be assumed to behave as-if they act upon the cavity only at a single radius, u_i is the acoustical velocity entering the cavity at the ith channel and w_i the radial channel width.

Again we use the equation given in equation 15 to provide us with an expression for the acoustical pressure in the compression cavity. We need not re-analyse the effect of the radiating membrane, instead we assume the pressure in the cavity can be described as the sum of two pressure contributions, one occurring due to the velocity of the membrane, p_m, and the other due to the velocity of air entering and leaving the cavity at the exit channels, p_c.

$$p = p_m + p_c \tag{22}$$

We already have the result for p_m.

$$p_m = \frac{\rho_o c_o^2}{V} \frac{S u_o}{j\omega} \tag{23}$$

And p_c is evaluated using the same summation of eigenfunctions as was applied above.

$$p_c(r) = \sum_{n=0}^{\infty} \frac{j\omega \rho_o \Psi_n(r)}{V\left[k_n^2 - k^2\right]} \int_{\hat{r}=0}^{r_o} \Psi_n(\hat{r}) u_e(\hat{r}) dS \tag{24}$$

The integral in the expression can be easily evaluated from our definition of $u_e(r)$ in equation 21 and the sifting property of the dirac delta function.

$$\int_{\hat{r}=0}^{r_o} \Psi_n(\hat{r}) u_e(\hat{r}) dS = \sum_{i=1}^{N} \Psi_n(r_i) A_i u_i \tag{25}$$

A_i is the area of the ith channel entrance.

Inserting the result into equation 24 we have a relatively compact expression for the part of the cavity pressure excited by motion of air in the channel entrances.

$$p_c(r) = \sum_{n=0}^{\infty} \frac{j\omega \rho_o \Psi_n(r)}{V\left[k_n^2 - k^2\right]} \sum_{i=1}^{N} \Psi_n(r_i) A_i u_i \tag{26}$$

This can be combined with the membrane excited pressure to give the full cavity pressure from both channel and membrane excitation.

$$p(r)=\frac{\rho_o c_o^2}{V}\frac{S u_o}{j\omega}+\sum_{n=0}^{\infty}\frac{j\omega\rho_o \Psi_n(r)}{V\left[k_n^2-k^2\right]}\sum_{i=1}^{N}\Psi_n(r_i)A_i u_i \tag{27}$$

I.ii. Suppression Of Modal Excitation By Channel Arrangement

We now use equation 27 as a starting point to derive a channel geometry to minimise the excitation of the acoustical modes in the compression cavity. Extracting the n=0 term from the summation in 27 we separate the excitation of the zeroth mode from the other excitation, as shown below.

$$p(r)=\frac{\rho_o c_o^2}{j\omega V}\left(S u_o+\sum_{i=1}^{N}A_i u_i\right)+\sum_{n=1}^{\infty}\frac{j\omega\rho_o \Psi_n(r)}{V\left[k_n^2-k^2\right]}\sum_{i=1}^{N}\Psi_n(r_i)A_i u_i \tag{28}$$

This is a very useful manipulation as it separates the desired lumped behaviour from the undesirable higher order excitation. From this it is clear that in order to suppress excitation of the mth mode we require that

$$\sum_{i=1}^{N}\Psi_m(r_i)A_i u_i=0 \tag{29}$$

The approach taken by Smith is to make the assumption that the velocities in the channel entrances are identical. This simplifies the condition to

$$\sum_{i=1}^{N}\Psi_m(r_i)A_i=0 \tag{30}$$

This is a fair simplification and can be justified as follows. Firstly, assuming that the final design will provide good suppression of the modes in the compression cavity, we expect the compression driver to behave as described by the zeroth mode terms in equation 28. In this case, the pressure in the cavity is simply related to the total volume velocity at the chamber walls.

$$p(r)=\frac{\rho_o c_o^2}{j\omega V}\left(S u_o+\sum_{i=1}^{N}A_i u_i\right) \tag{31}$$

The velocities at the channel entrances can be related to the cavity pressure at the channel entrance by the acoustic impedance of the channel.

$$u_i=\frac{p}{z_i} \tag{32}$$

Smith then also assumes that the acoustical impedance at the each of the channels is identical in order to make the simplification that the channel entrance velocities are identical.

The condition set in equation 30 can be met for specific modes by careful selection of the channel positions, r_i, and entrance areas, A_i. If we have N channels it is possible to meet the condition for N modes. Logically we choose the lowest N modes of the compression chamber for two reasons: firstly to extend the bandwidth of the driver as high as possible in frequency and secondly because our simplifications above require that the zeroth modal behaviour is seen where these modes are suppressed.

The condition is a set of N simultaneous equations which can be solved by writing them in matrix form as shown below.

$$\begin{bmatrix} \Psi_1(r_1) & \Psi_1(r_2) & \Psi_1(r_3) & \cdots & \Psi_1(r_N) \\ \Psi_2(r_1) & \Psi_2(r_2) & \Psi_2(r_3) & \cdots & \Psi_2(r_N) \\ \Psi_3(r_1) & \Psi_3(r_2) & \Psi_3(r_3) & \cdots & \Psi_3(r_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Psi_N(r_1) & \Psi_N(r_2) & \Psi_N(r_3) & \cdots & \Psi_N(r_N) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (33)$$

From here we see that there are non-trivial solutions for the channel areas when the left-hand matrix has a determinant of zero. At first glance it appears to be quite difficult to construct the matrix of eigenfunction contributions on the left to meet this property, however, we can manipulate the condition into a form which is easier to resolve.

Firstly we move one column of the matrix, corresponding to the first term of the summation in equation 30, to the right hand side of the expression. We then divide by A_1 so that we now have a set of equations with area ratios as our variables.

$$\begin{bmatrix} \Psi_{\backslash}(r_2) & \Psi_1(r_3) & \cdots & \Psi_1(r_N) \\ \Psi_2(r_2) & \Psi_2(r_3) & \cdots & \Psi_2(r_N) \\ \Psi_3(r_2) & \Psi_3(r_3) & \cdots & \Psi_3(r_N) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_N(r_2) & \Psi_N(r_3) & \cdots & \Psi_N(r_N) \end{bmatrix} \begin{bmatrix} A_2/A_1 \\ A_3/A_1 \\ \vdots \\ A_N/A_1 \end{bmatrix} = \begin{bmatrix} -\Psi_1(r_1) \\ -\Psi_2(r_1) \\ -\Psi_3(r_1) \\ \vdots \\ -\Psi_N(r_1) \end{bmatrix} \qquad (34)$$

The matrix on the left hand side is no-longer square, it is N by N-1. In order to make the matrix square and the system of equations soluble we must eliminate a row from the matrix. This can be easily done for the Nth row as we can choose the positions of $r_{\backslash,} r_{2,} r_3 \ldots$ to coincide with the N nodal positions of the Nth mode, thus we ensure $\Psi_N(r_i)=0$ and we have 35. Which is easily soluble by inversion of the matrix.

$$\begin{bmatrix} \Psi_1(r_2) & \Psi_1(r_3) & \cdots & \Psi_1(r_N) \\ \Psi_2(r_2) & \Psi_2(r_3) & \cdots & \Psi_2(r_N) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{N-1}(r_2) & \Psi_{N-1}(r_3) & \cdots & \Psi_{N-1}(r_N) \end{bmatrix} \begin{bmatrix} A_2/A_1 \\ A_3/A_1 \\ \vdots \\ A_N/A_1 \end{bmatrix} = \begin{bmatrix} -\Psi_1(r_1) \\ -\Psi_2(r_1) \\ \vdots \\ -\Psi_{N-1}(r_1) \end{bmatrix} \qquad (35)$$

For example in the case of a three channel phase plug we choose.

$$
\begin{aligned}
r_1 &= 0.238\, r_o \\
r_2 &= 0.543\, r_0 \\
r_3 &= 0.853\, r_o
\end{aligned}
\tag{36}
$$

Solving the simultaneous equations as described above gives the corresponding area ratios

$$
\begin{aligned}
\frac{A_{\text{٣}}}{A_1} &= \text{٣.٨١٧} \\
\frac{A_2}{A_{\text{١}}} &= 2.3386
\end{aligned}
\tag{37}
$$

Which can be equivalently written as channel width ratios

$$
\begin{aligned}
\frac{w_3}{w_1} &= 1.065 \\
\frac{w_2}{w_1} &= 1.025
\end{aligned}
\tag{38}
$$

This result is a repetition of that demonstrated by [1].